Quantization based Optimization : Alternative Stochastic Approximation of Global Optimization

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Abstract

In this paper, we propose a global optimization scheme based on quantizing the energy level of an objective function in an NP-hard problem. According to the white noise hypothesis for the quantization error with dense and uniform distribution, we can regard the quantization error as an i.i.d. white noise. Additionally, the stochastic analysis shows that the proposed algorithm converges weakly under the condition satisfying the Lipschitz continuity only, instead of local convergence properties such as the Hessian constraint of the objective function. It leads that the proposed algorithm ensures global optimization by Laplace's condition. Numerical experiments show that the proposed algorithm performs better in NP-hard optimization problems such as the traveling salesman problem(TSP) than the conventional learning method.

1. Introduction

Finding the global optimum in the non-deterministic polynomial hardness problem(NP-Hardness problem) such as the TSP has been a crucial research theme ([3, 6, 10, 20, 28, 34, 36, 38]). Since Kirkpatrick et al. [29] presented the simulated annealing (SA) in 1980s, researchers have developed and applied various heuristic algorithms to combinatorial optimization problems, including NP-hardness problems ([2, 11, 12, 14, 15, 17, 18, 26, 27, 35, 37, 39, 42, 43]). Despite regarding such heuristic algorithms as alternatives of a stochastic optimization technique, the fundamental dynamics of some algorithms are still unclear([1, 4, 8, 22]). Those unclear dynamics yields some problems such as selection of suitable hyper-parameters for the optimization performance[19].

In contrast to the conventional natural phenomenon-based optimization algorithms, we propose the quantization-based optimization algorithm with a monotonically increasing quantization resolution in this paper. As shown in [5, 31, 41], the main research topic for quantization has been minimizing the effect of quantization error in signal processing, and this approach is the same for artificial intelligence and machine learning, as shown in [9, 25, 40]. However, if the distribution of quantization error is sufficiently dense and follows a uniform distribution, we can let the quantization error be a white noise as presented in Gray and Neuhoff [16]'s paper. Additionally, [23] proved that it is an i.i.d white noise if components of a dense quantization error vector are asymptotically pairwise independent and distributed uniformly. This property is known as the quantization error's white noise hypothesis(WNH). Accordingly, we can constitute a proper stochastic global optimization algorithm based on a quantization error. We provide stochastic analysis to prove the proposed algorithm's weak convergence for global optimization based on Laplace's theorem presented by [7, 13, 32, 33], and we apply the proposed algorithm to the TSP to verify the algorithm's validity. Especially in the TSP with many cities, the proposed algorithm shows better optimization performance than the SA and the quantum annealing (QA) algorithm.

2. Preliminaries

2.1. Definitions of quantization

Before illustrating the proposed algorithm, we set the following definitions and assumptions.

Definition 1 *For $f \in \mathbf{R}$, we define the quantization of f as follows:*

$$f^Q \triangleq \frac{1}{Q_p} \lfloor Q_p \cdot (f + 0.5 \cdot Q_p^{-1}) \rfloor = \frac{1}{Q_p} (Q_p \cdot f + \varepsilon) = f + \varepsilon Q_p^{-1}, \quad f^Q \in \mathbf{Q} \tag{1}$$

, where $\lfloor f \rfloor \in \mathbf{Z}$ is the floor function such that $\lfloor f \rfloor \leq f$ for all $f \in \mathbf{R}$, $Q_p \in \mathbf{Q}^+$ is the quantization parameter, and $\varepsilon \in \mathbf{R}$ is the quantization error.

Definition 2 *We define the quantization parameter $Q_p \in \mathbf{Q}^+$ to be a monotone increasing function $Q_p : \mathbf{R}^{++} \mapsto \mathbf{Z}^+$ such that*

$$Q_p(t) = \eta \cdot b^{\bar{h}(t)} \tag{2}$$

, where $\eta \in \mathbf{Q}^{++}$ is the fixed constant parameter of the quantization parameter, b is the base, and $\bar{h} : \mathbf{R}^{++} \mapsto \mathbf{Z}^+$ is the power function such that $\bar{h}(t) \uparrow \infty$ as $t \to \infty$.

Assumption 1 *For a numerical sequence $\{f(t)\}_{t=0}^{\infty}$ where each $f(t) \in \mathbf{R}^+ \ \forall t > 0$, suppose that $f(t)$ is defined on a dense topology space. By the WNH and the equation (1), the quantization error ε_t corresponding to $t > 0$ is an i.i.d. white noise defined on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P}_\varepsilon)$.*

Under assumption 1, we can regard the sequence of the f_t^Q corresponding to $f(t)$ is a stochastic process $\{f_t^Q\}_{t=0}^{\infty}$. As a next step, to analyze the properties of the stochastic process $\{f_t^Q\}_{t=0}^{\infty}$, we calculate the mean and the variance of the quantization error.

Theorem 1 *If the quantization error $\varepsilon_t \in \mathbf{R}^n$ satisfying the WNH, the mean and the variance of the quantization error at $t > 0$ is*

$$\forall \varepsilon_t \in \mathbf{R}, \quad \mathbb{E}_{\mathcal{F}_t} Q_p(t) \varepsilon_t = 0, \quad \mathbb{E}_{\mathcal{F}_t} Q_p^{-2}(t) \varepsilon_t^2 = Q_p^{-2}(t) \cdot \mathbb{E}_{\mathcal{F}_t} \varepsilon_t^2 = \frac{1}{12 \cdot Q_p^2(t)}. \tag{3}$$

To discuss the main algorithm, we consider the optimization problem for an objective function f such that

$$\text{minimize} \quad f : \mathbf{R}^n \mapsto \mathbf{R}^+. \tag{4}$$

In various combinatorial optimization problems, we deal with an actual input represented as $x^r \in [0, 1]^m$. Thus, we suppose that There exists a proper transformation from a binary input to a proper real vector space such that $\mathcal{T} : [0, 1]^m \to \mathcal{X} \subseteq \mathbf{R}^n$., where $\mathcal{X}$ is the virtual domain of the objective function f. Under this transformation assumption, we assume that the $f \in C^\infty$ fulfills the Lipschitz continuity as follows:

Assumption 2 *For $x_t \in B^o(x^*, \rho)$, there exist a positive value L w.r.t. a scalar field $f(x) : \mathbf{R}^n \to \mathbf{R}$ such that*

$$\|f(x_t) - f(x^*)\| \leq L \|x_t - x^*\|, \quad \forall t > t_0, \tag{5}$$

where $B^o(x^*, \rho)$ is an open ball $B^o(x^*, \rho) = \{x | \|x - x^*\| < \rho\}$ for all $\rho \in \mathbf{R}^{++}$, and $x^* \in \mathbf{R}^n$ is the globally optimal point.

Algorithm 1: Blind Random Search (BRS) with the proposed quantization scheme

Input: Objective function $f(x) \in \mathbf{R}^+$
Output: $x_{opt},\ f(x_{opt})$
Data: $x \in \mathbf{R}^n$
Initialization
$t \leftarrow 0$ and $\bar{h}(0) \leftarrow 0$
Set initial candidate x_0 and $x_{opt} \leftarrow x_0$
Compute the initial objective function $f(x_0)$
Set $b = 2$ and $\eta = b^{-\lfloor \log_b(f(x_0)+1) \rfloor},\ Q_p \leftarrow \eta$
$f_{opt}^Q \leftarrow \frac{1}{Q_p} \lfloor Q_p \cdot (f + 0.5 \cdot Q_p^{-1}) \rfloor$
while *Stopping condition is satisfied* **do**
  Set $t \leftarrow t + 1$
  Select x_t randomly and compute $f(x_t)$
  $f^Q \leftarrow \frac{1}{Q_p} \lfloor Q_p \cdot (f + 0.5 \cdot Q_p^{-1}) \rfloor$
  if $f^Q \leq f_{opt}^Q$ **then**
    $x_{opt} \leftarrow x_t$
    $\bar{h}(t) \leftarrow \bar{h}(t) + 1,\ Q_p \leftarrow \eta \cdot b^{\bar{h}(t)}$
    $f_{opt}^Q \leftarrow \frac{1}{Q_p} \lfloor Q_p \cdot (f + 0.5 \cdot Q_p^{-1}) \rfloor$
  end
end

2.2. Primitive algorithm

As the most elementary implementation, we apply the proposed quantization scheme to the blind random search(BRS) algorithm.

First, as shown in Algorithm 1, we randomly select a input point x_t and we quantize the value of objective function $f(x_t)$ such that $f^Q(x_t)$ with the quantization parameter $Q_p(t-1)$. Comparing both quantization values $f^Q(\bar{x}_{t-1})$ and $f^Q(x_t)$, if $f^Q(\bar{x}_{t-1})$ is larger than or **equal** to the $f^Q(x_t)$, then we set x_t to be the optimal value and substitute $\bar{x}_t$ to the x_t. Following this procedure, we update the quantization parameter as $Q_p(t-1)$ with increasing the power function $\bar{h}(t)$ defined in (2). We denote it as the re-quantization. Since we update the quantization parameter, the quantization value of $f^Q(x_t)$ is re-quantized with $Q_p(t)$. Consecutively, we select another input point as a part of a blind random search.

Furthermore, we propose a simple initialization of the quantization parameter to implement the BRS using the proposed scheme. We want the transition probability of the initial state $\mathbb{P}(x_1|x_0)$ to be a high probability such as $\mathbb{P}(x_1|x_0) = 1$. Therefore, the quantization of all the other objective function value $f^Q(x_1)\forall x_1 \neq x_0$ should be lower than the quantization of the initial objective function possibly. For this purpose, we set the initial parameter of the quantization parameter η as represented as following theorem:

Theorem 2 *Suppose that the initial value of a given objective function $f(x_0) \in \mathbf{R}$ is $\sup_{x\in\mathbf{R}} f(x)$. The transition probability to the next step, $\mathbb{P}(x_1|x_0)$, yielded by the proposed algorithm is one when the initial parameter $\eta \in \mathbf{Q}^+$ satisfies the following equation:*

$$\eta = b^{-\lfloor log_b(f(x_0)+1) \rfloor} \tag{6}$$

, where b is a base in definition 2 for $Q_p(t)$.

3. Analysis of the proposed algorithm

3.1. Fundamental dynamics of the proposed algorithm

Let a subset of the virtual domain $\mathcal{X} \subseteq \mathbf{R}^n$ such that $L^Q(t) \triangleq \{x_t | f(x) - f^Q(\bar{x}_t) \leq 0, Q_p(t)\}$. Under the procedure in Algorithm 1 and the definition of the subset $L^Q(t)$, we note that the proposed algorithm can yield the following containment relationship between subsets:

$$\exists t > t_0,\ L^Q(t) \supseteq L^Q(t+1) \cdots \supseteq L^Q(t+k) \tag{7}$$

The above equation can lead to the measure of $L^Q(t)$ being proportion to $Q_p^{-1}(t)$ by Lipschitz continuous represented in Assumption 2. In addition, since $Q_p^{-1}(t)$ decreases monotonically by definition 2, we can obtain the following inequalities about the measure of each subset:

$$\exists t > t_0,\ m(L^Q(t)) \geq m(L^Q(t+1)) \cdots \geq m(L^Q(t+k)) \tag{8}$$

Suppose that there exists a unique optimizer x^* such that $\forall x \in \mathcal{X},\ f(x^*) \leq f(x) \leq f^Q(x)$. If (8) brings $\lim_{t\uparrow\infty} m(L^Q(t+k)) = 0$, we can note that $f^Q(x) \to f(x)$. Accordingly, by the above assumption of unique optimizer, we can obtain $f^Q(x) \to f(x^*)$ intuitively.

To prove the above consideration, we set the following assumption.

Assumption 3 *The power summation to the base $b^{\bar{h}(t+k)}$ is bounded such that*

$$\lim_{k\to\infty} \sum_{k=0}^{n} b^{-\bar{h}(t+k)} = \bar{b}(t) < \infty, \quad \bar{b}(t) \downarrow 0 \text{ as } t \uparrow \infty \tag{9}$$

Under the assumption 3, we can establish the following theorem

Theorem 3 *For a large $k > n_0$, if the proposed algorithm provides the sufficiently finite resolution for f^Q such that*

$$f^Q(x_{t+k}) - f^Q(x_{t+k+1}) = Q_p(t+k)^{-1} \tag{10}$$

, for all $x_t \in \mathbf{R}^n$ and $t > 0$, there exists $n < n_0$ satisfying the following

$$\|f(x_{t+n}) - f(x_{t+n+1})\| \geq \|f(x_{t+k}) - f(x^*)\|. \tag{11}$$

For the stochastic analysis of the proposed algorithm, we can obtain the following lemma associated with the difference of quantization errors to the quantized objective functions.

Lemma 4 *Suppose that there exist two equal quantized objective functions for two distinguished inputs $x_t, x_{t+1} \in \mathbf{R}^n$ such that $f^Q(x_t) = f^Q(x_{t+1})$. Under this condition, the quantization error $\bar{\varepsilon}_t Q_p^{-1}(t)$ of $f^Q(x_t) - f^Q(x_{t+1})$ is evaluated as follows:*

$$\bar{\varepsilon}_t Q_p^{-1} = (\varepsilon_{t+1} - \varepsilon_t) \cdot (x_{t+1} - x_t) \cdot v_t \cdot \tilde{Q}_p^{-1} \tag{12}$$

, where v_t is a normalized vector defined as $v_t = \frac{x_{t+1}-x_t}{\|x_{t+1}-x_t\|}$ and $\tilde{Q}_p(t)$ is a scaled quantization parameter to $Q_p(t)$ with a constant value $C \in \mathbf{R}^+$ such that

$$\tilde{Q}_p^{-1}(t) = C \cdot b^{\bar{h}(t)}. \tag{13}$$

With the above theorem and lemma, we can establish the stochastic differential equation(SDE) for the proposed algorithm as follows:

Theorem 5 *For a given objective function $f(x_t) \in \mathbf{R}$, suppose that there exist the quantized objective functions $f^Q(x_t),\ f^Q(x_{t+1})$ at a current state x_t and the following state x_{t+1} such that $f^Q(x_t) \geq f^Q(x_{t+1})$, for all $x_{t+1} \neq x_t$; we can obtain the differential equation of the state transition as follows:*

$$dX_t = -\nabla_x f(X_t)dt + \sqrt{C_q} \cdot Q_p^{-1}(t) dW_t \tag{14}$$

where W_t is a standard Wiener process, which has a zero mean and variance with one, X_t is a random variable corresponding to x_t, and $C_q \in \mathbf{R}$ is a constant value.

3.2. Weak convergence of the proposed algorithm

The equation (14) is the typical Langevine SDE, so we can expect that the transition probability yielded by the proposed algorithm follows Gibb's distribution based on a Gaussian function. Additionally, we note that the proposed algorithm involves the hill-climbing effect brought by the Wiener process dW_t; hence the proposed algorithm is robust to local minima [32, 33]. However, an asymptotic analysis of the Hilbert space always represents the possibilities of divergence in the optimization algorithm containing the hill-climbing property. Therefore, we show a global optimization of the algorithm including the hill-climbing, which is robust to local minima, so that we prove the convergence of the transition probability yielded by the proposed algorithm to global optimum. We denotes this convergence as a weak convergence. Particularly, the proof of weak convergence to the transition probability represented with Gibbs's Distribution is relatively clear based on the Laplace theorem. In the proposed algorithm, as shown in (14), the variance of the transition probability is in proportion to the inverse of the quantization parameter $Q_p^{-1}(t)$. The inverse of the quantization parameter is a monotone decreasing function to time t as represented in Definition 2, and the limit of the summation to time is finite as shown in Assumption 3. Consequently, we can expect that the proposed algorithm fulfills Laplace's theorem([7, 13, 21, 32]), and we can prove the weak convergence as follows:

Theorem 6 *If the dynamics of the state transition by the proposed algorithm follow (14), the state x_t weakly converges to the global minimum when the quantization parameter decreases to the following schedule:*

$$\inf_{t \geq 0} Q_p^{-1}(t) = \frac{\sqrt{12} \cdot C}{\log(t+2)}, \quad C \in \mathbf{R}^+,\ C \gg 0 \tag{15}$$

With the assumption of an objective function's Lipschitz continuous property, we can prove Theorem 6 without any convex assumptions. Another property shown by Theorem 6 is that the proposed primitive algorithm contains more strong convergence conditions than Theorem 6 represents. In addition, since Q_p^{-1} is not a rational number, the implementation of the proposed algorithm may be elusive. Therefore, letting (15) be an upper bound and another equation be a lower bound, we can set $\bar{h}(t)$ as the following theorem for the global convergence and implementation of the proposed algorithm.

Theorem 7 *Suppose that there exists an integer valued annealing schedule $\sigma(t) \in \mathbf{Z}^+$ such that $\sigma(t) \geq \inf \sigma(t) \triangleq c/\log(t+2)$. If the power function $\bar{h}(t)$ of the quantization parameter $Q_p^{-1}(t)$ fulfills the following condition, the proposed algorithm weakly converges to the global optimum.*

$$\log_b \left(C_0 \cdot b^{-\frac{2\beta}{t+2}} \cdot \inf \sigma(t) \right) \leq \bar{h}(t) \leq \log_b \left(C_1 \log(t+2) \right) \tag{16}$$

, where $C_0 \equiv \eta \sqrt{C_q}$ and $C_1 \equiv \sqrt{C_q}\eta / C$.

Theorem 7 illustrates that if the algorithm controls the power function $\bar{h}(t)$ for the quantization under the condition in theorem 7, we can find the global optimum with weak convergence property.

4. Simulation Results

To verify the optimization performance of the proposed algorithm for combinatorial optimization problems, including NP-hardness, we accomplish the TSP simulation for 100 cities located in the

Table 1: Simulation Results to TSP for 100 cities

Criterion	Simulated Annealing	Quantum Annealing	Proposed
Average Minimum Cost	1729.50	1721.07	1648.26
Improvement Ratio to the Initial setting	19.90%	20.29%	23.67 %

Table 2: Simulation Results to TSP beyond 100 cities

Number of Cities	Nearest Neighbor(Initial)	Simulated Annealing	Quantum Annealing	Proposed	Improve Ratio
100	2159.27	1727.44	1729.69	1706.53	20.96
125	2297.86	2027.52	2028.2	1923.65	16.28
150	2497.65	2255.15	2252.82	2032.21	18.63
175	2380.52	2380.52	2380.29	2147.17	9.80
200	2769.73	2769.34	2769.42	2366.72	14.55

2-dimensional squared space with the range $[0, 200]$. We use the OPT-2 algorithm, which is the selection method of cities in TSP for the cost evaluation presented by [24]. The OPT-2 algorithm is one of the transform functions for real binary input space to a virtual real vector space such as $\mathcal{T} : \{0, 1\}^m \to \mathbf{R}^n$, where m is the number of cities minus 1, n is a virtual dimension of the virtual space. Using such transformation, we can assume that an objective function fulfills Lipschitz continuity in that the OPT-2 algorithm changes the location of only two cities[13]. In all attempts, we use a fixed location of cities to guarantee the generality of the simulation as possible. Moreover, to guarantee an objective optimization performance for all algorithms in simulation, we set an initial route for each city with the Nearest Neighbor algorithm for TSP, and we set the initial route as a start Hamiltonian H_0 for quantum annealing. The simulation result in Table 1 shows that the average optimization performance of the proposed algorithm is superior to that of classical annealing and quantum annealing.

Furthermore, we test the optimization performance of algorithms to TSP beyond 100 cities. As well known, the difficulties to TSP beyond 100 cities increase dramatically. For instance, the possible number of routes in TSP from 100 cities to 110 cities increase approximately 10^20 times (from 9.33×10^{157} to 1.58×10^{178}). Such increasing difficulties in TSP cause heavy computational times for operation and failure of optimization. The simulation result in Table 2 represents that the proposed algorithm can find a feasible solution even when the number of cities increases to 200, whereas other algorithms fail to find a better solution than the Nearest Neighborhood method does.

5. Conclusion

We present a quantization-based optimization scheme with an increase in the quantization resolution to optimize an objective function globally. Provided stochastic analysis brings the SDE describing the dynamics of the proposed algorithm. Using the SDE and feasible assumptions, we present the analysis for weak convergence of the proposed algorithm enabling global optimization. The proposed algorithm is based on the mathematical feature of quantization error, whereas other heuristic algorithms simulate natural phenomena. Consequently, we expect to develop an alternative global optimization methodology by numerical analysis based on number theory. In future work, we will research an effective iterative difference learning equation based on a quantized optimization scheme for a continuous function to apply it to general machine learning and artificial intelligence algorithms.

Appendix A. Introduction

We set notations, proof of lemmas and theorems and more detailed information about the simulation in the manuscript to the following sections.

Appendix B. Notations

- $\mathbf{R}^n$ The n-dimensional space with real numbers
- $\mathbf{R}$ $\mathbf{R}^n|_{n=1}$
- $\mathbf{R}[\alpha, \beta]$ $\{x \in \mathbf{R} | \alpha \leq x \leq \beta, \ \alpha, \beta \in \mathbf{R}\}$
- $\mathbf{R}(\alpha, \beta]$ $\{x \in \mathbf{R} | \alpha < x \leq \beta, \ \alpha, \beta \in \mathbf{R}\}$
- $\mathbf{R}[\alpha, \beta)$ $\{x \in \mathbf{R} | \alpha \leq x < \beta, \ \alpha, \beta \in \mathbf{R}\}$
- $\mathbf{R}(\alpha, \beta)$ $\{x \in \mathbf{R} | \alpha < x < \beta, \ \alpha, \beta \in \mathbf{R}\}$
- $\mathbf{Q}^n$ The n-dimensional space with rational numbers
- $\mathbf{Q}$ $\mathbf{Q}^n|_{n=1}$
- $\mathbf{Z}$ The 1-dimensional space with integers.
- $\mathbf{N}$ The 1-dimensional space with natural numbers.
- $\mathbf{R}^+$ $\{x | x \geq 0, \ x \in \mathbf{R}\}$
- $\mathbf{R}^{++}$ $\{x | x > 0, \ x \in \mathbf{R}\}$
- $\mathbf{Q}^+$ $\{x | x \geq 0, \ x \in \mathbf{Q}\}$
- $\mathbf{Q}^{++}$ $\{x | x > 0, \ x \in \mathbf{Q}\}$
- $\mathbf{Z}^+$ $\{x | x \geq 0, \ x \in \mathbf{Z}\}$
- $\mathbf{Z}^{++}$ $\{x | x > 0, \ x \in \mathbf{Z}\}$, $\mathbf{Z}^{++}$ is equal to $\mathbf{N}$.
- $\lfloor x \rfloor$ $\max\{y \in \mathbf{Z} | y \leq x, \forall x \in \mathbf{R}\}$
- $\lceil x \rceil$ $\min\{y \in \mathbf{Z} | y \geq x, \forall x \in \mathbf{R}\}$

Appendix C. Auxiliary Lamma

We use the following lemma to prove the theorems represented in the next chapters.

Lemma : Auxiliary 1 *For all* $x \in \mathbf{R}$*,*

$$(1 - x) \leq \exp(-x). \tag{17}$$

Proof By definition of the exponent, we write the exponential function as following fundamental series :

$$\exp(-x) = \sum_{i=0}^{\infty} \frac{1}{i!}(-1)^i x^i = \sum_{k=0}^{\infty} \left(\frac{1}{2k!} x^{2k} - \frac{1}{(2k+1)!} x^{2k+1} \right). \tag{18}$$

Let u_k as follows:

$$u_k = \frac{1}{2k!} x^{2k} \left(1 - \frac{1}{2k+1} x \right) \tag{19}$$

, then we can rewrite the series of exponent such that

$$\exp(-x) = u_0 + \sum_{k=1}^{\infty} u_k. \tag{20}$$

For all $k > 0$, since each u_k is positive, we have

$$1 - x = u_0 \leq u_0 + \sum_{k=0}^{\infty} u_k. \tag{21}$$

Alternatively, we can prove the lemma with differentiation. Let $g(x) = (1-x) - \exp(-x)$. Differentiating $g(x)$ to x, we get

$$\frac{dg}{dx}(x) = -1 + exp(-x), \; \frac{d^2 g}{dx^2} = -\exp(-x) \tag{22}$$

We note that $g(x)$ is a concave function since $\frac{d^2 g}{dx^2} < 0, \; \forall x \in \mathbf{R}$. In addition, the maximum of $g(x)$ is zero at $x = 0$ from which $\frac{dg}{dx}(x) = -1 + exp(-x) = 0$. Therefore, since $g(x) \leq 0$, it fulfills the Lemma. ■

Appendix D. Proofs of the theorems in section 2

D.1. Proof of theorem 1

Theorem 1 *If the quantization error $\varepsilon_t \in \mathbf{R}^n$ satisfying the WNH, the mean and the variance of the quantization error at $t > 0$ is*

$$\forall \varepsilon_t \in \mathbf{R}, \quad \mathbb{E}_{\mathcal{F}_t} Q_p(t) \varepsilon_t = 0, \quad \mathbb{E}_{\mathcal{F}_t} Q_p^{-2}(t) \varepsilon_t^2 = Q_p^{-2}(t) \cdot \mathbb{E}_{\mathcal{F}_t} \varepsilon_t^2 = \frac{1}{12 \cdot Q_p^2(t)}. \tag{23}$$

Proof The theorem is explicit according to the WNH. Let Δ be the brief notation of $\varepsilon_t Q_p^{-1}(t)$. According to Jiménez et al. [23], ε_t is uniformly distributed in $[-Q_p^{-1}(t), Q_p^{-1}(t))$ under the WNH and Definition 1. Therefore we can obtain the expectation value of $\Delta = \varepsilon_t Q_p^{-1}(t)$ as follows:

$$\mathbb{E}_{\mathcal{F}_t} \Delta = \int_{-Q_p^{-1}(t)/2}^{Q_p^{-1}(t)/2} \Delta \mathbb{P}_\varepsilon d\Delta = \frac{1}{Q_p^{-1}(t)} \cdot \int_{-Q_p^{-1}(t)/2}^{Q_p^{-1}(t)/2} \Delta d\Delta = \frac{1}{2Q_p^{-1}(t)} \left(\frac{Q_p^{-1}(t)^2}{2^2} - \frac{Q_p^{-1}(t)^2}{(-2)^2} \right) = 0. \tag{24}$$

In a similar way, we can obtain the variance such that

$$\mathbb{E}_{\mathcal{F}_t}\Delta^2 = \frac{1}{Q_p^{-1}(t)} \int_{-Q_p^{-1}(t)/2}^{Q_p^{-1}(t)/2} \Delta^2 d\Delta = \frac{1}{Q_p^{-1}(t)} \cdot \frac{1}{3} \left(\frac{Q_p^{-1}(t)^3}{8} - \frac{-Q_p^{-1}(t)^3}{8} \right) = \frac{1}{12 \cdot Q_p^2(t)} \quad (25)$$

From the WNH, the square of ε_t is one, so that we obtain the result of the theorem. ■

D.2. Proof of theorem 2

Theorem 2 *Suppose that the initial value of a given objective function $f(x_0) \in \mathbf{R}$ is $\sup_{x \in \mathbf{R}} f(x)$, then the transition probability to a next step $\mathbb{P}(x_1|x_0)$ lead by the proposed algorithm is one when the initial parameter $\eta \in \mathbf{Q}^+$ of the Q_p satisfying*

$$\eta = b^{-\lfloor log_b(f(x_0)+1) \rfloor} \quad (26)$$

, where b is a base in definition 2 for Q_p defined .

Proof By assumption, we can set the following inequality for all $x_1 \neq x_0$

$$f(x_0) + Q_p^{-1}(0) \geq f(x_1) + Q_p^{-1}(1) \quad (27)$$

By definition of the quantization parameter Q_p, $Q_p(0) = \eta b^0 = \eta$ and $Q_p(1) = \eta b^{-1}$, thus

$$f(x_0) + \eta^{-1} \geq f(x_1) + \eta^{-1} b \implies f(x_0) - f(x_1) \geq \eta^{-1}(b-1). \quad (28)$$

Suppose that η is a power of b, i.e., $\eta = b^k$, where $k \in \mathbf{Z}^+$. Substitute η with the power of b such that

$$\begin{aligned} f(x_0) - f(x_1) \geq b^{-k}(b-1) &\implies \frac{f(x_0) - f(x)}{b-1} \geq b^{-k} \\ \implies -\log_b \frac{f(x_0) - f(x_1)}{b-1} \leq k &\implies k \geq \log_b(b-1) - \log_b(f(x_0) - f(x)). \end{aligned} \quad (29)$$

Since $\log_b(b-1) \geq 0$ and $\log_b(f(x_0) - f(x_1)) \geq \log_b f(x_0)$ for all $x_1 \neq x_0$, we obtain

$$k \geq \log_b(b-1) - \log_b(f(x_0) - f(x)) > -1 - \log_b f(x_0) \geq -\lfloor 1 + \log_b f(x_0) \rfloor \quad (30)$$

Therefore, if $f(x_0) \in \mathbf{R}$ is $\sup_{x \in \mathbf{R}} f(x)$, thus the initial transition probability is one, we can set the initial value of quantization parameter $\eta = Q_p(0)$ to be

$$\eta = b^{-\lfloor log_b(f(x_0)+1 \rfloor} \quad (31)$$

■

Appendix E. Proof of lemma and theorem in section 3

E.1. the proof of theorem 3

Theorem 3 *For a large $k > n_0$, if the proposed algorithm provides the sufficiently finite resolution for f^Q such that*

$$f^Q(x_{t+k}) - f^Q(x_{t+k+1}) = Q_p(t+k)^{-1} \tag{32}$$

, for all $x_t \in \mathbf{R}^n$ and $t > 0$, there exists $n < n_0$ satisfying the following

$$\|f(x_{t+n}) - f(x_{t+n+1})\| \geq \|f(x_{t+k}) - f(x^*)\|. \tag{33}$$

Proof Assume that $f^Q(x^*) = f(x^*)$, and $f^Q(x) \neq f^Q(y)$ for all $x, y \in \mathbf{R}^n$ and $x \neq y$. From the definition of the algorithm, we note that the infimum of the difference between $f^Q(x)$ and $f^Q(y)$ is $Q_p(\tau)^{-1}$ when $f^Q(x)$ and $f^Q(y)$ are not equal. Thus, we can obtain

$$f^Q(x_s) - f^Q(x_{s+1}) \geq Q_p(s)^{-1} = \eta^{-1} \cdot b^{-\bar{h}(s)}, \quad \forall b \in \mathbf{Z}(1, \infty) \tag{34}$$

, where $s \in \mathbf{Z}^+$. By assumption, for an positive real integer $\tau > s$, the difference $f^Q(x_\tau) - f^Q(x_{\tau+1})$ is equal to the each quantization step i.e. $f^Q(x_\tau) - f^Q(x_{\tau+1}) = \eta^{-1} \cdot b^{-\bar{h}(\tau)}$ Accordingly, (34) leads

$$\begin{aligned} f^Q(x_\tau) - f^Q(x^*) &= f^Q(x_\tau) - f^Q(x_{\tau+1}) + f^Q(x_{\tau+1}) - \cdots - f^Q(x_{\tau+n}) + f^Q(x_{\tau+n}) - f^Q(x^*) \\ &= \eta^{-1} \sum_{k=0}^{n-1} b^{-\bar{h}(\tau+k)} + f^Q(x_{\tau+n}) - f^Q(x^*). \end{aligned} \tag{35}$$

If we can find the optimal point at the step $\tau + n$, we can obtain the supremum of the bound to the difference $f^Q(x_{\tau+n}) - f^Q(x^*)$ as follows:

$$\begin{aligned} \sup \inf_{x_{\tau+n}} \|f^Q(x_{\tau+n}) - f^Q(x^*)\| &= \sup \inf_{x_{\tau+n}} \|f^Q(x_{\tau+n}) - f(x^*)\| \\ &= \sup \inf_{x_{\tau+n}} \|f(x^*) + \varepsilon Q_p^{-\bar{h}(\tau+n)} - f(x^*)\| \\ &= Q_p^{-\bar{h}(\tau+n)} = \eta^{-1} \cdot b^{-\bar{h}(\tau+n)}. \end{aligned} \tag{36}$$

Thus, we can obtain

$$\begin{aligned} f^Q(x_\tau) - f^Q(x^*) &\leq \eta^{-1} \sum_{k=0}^{n-1} b^{-\bar{h}(\tau+k)} + \eta^{-1} \cdot b^{-\bar{h}(\tau+n)} \\ &= \eta^{-1} \sum_{k=0}^{n} b^{-\bar{h}(\tau+k)} < \eta^{-1} \sum_{k=0}^{\infty} b^{-\bar{h}(\tau+k)} = \eta^{-1} \cdot \bar{b}(\tau). \end{aligned} \tag{37}$$

By assumption, since the $\bar{b}(t)$ is a monotone decreasing function with respect to t, there exists $\delta > 0$ such that $\delta > \bar{b}(\tau)$. Therefore, there exists $s > \tau$ such that

$$f^Q(x_s) - f^Q(x_{s+1}) \geq \eta^{-1} \cdot b^{-\bar{h}(s)} \geq \eta^{-1} \cdot \delta > \eta^{-1} \cdot \bar{b}(\tau) > f^Q(x_\tau) - f^Q(x^*). \tag{38}$$

■

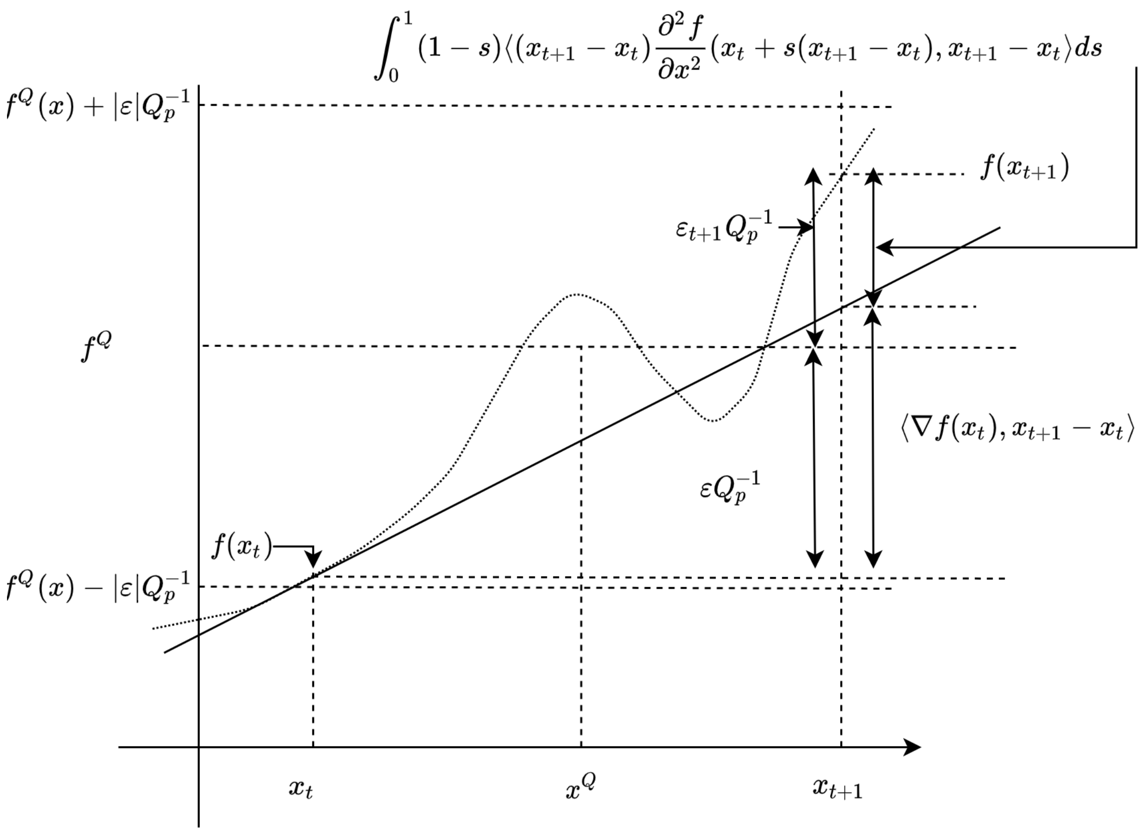


Figure 1: Conceptual diagram for Lemma 4. In an equal quantization level, we can dismiss correct value of $f(x)$ since $f^Q(x) = f^Q(y), \forall x \neq y$.Thus, we can let $f(x)$ as a simple low-order function such as the first-order function, within an equal quantization level instead of the correct $f(x)$

E.2. the proof of Lemma 4

Lemma 4 *Suppose that there exist two equal quantized objective functions for two distingushed inputs* $x_t, x_{t+1} \in \mathbf{R}^n$ *such that* $f^Q(x_t) = f^Q(x_{t+1})$. *Under this condition, the quantization error* $\bar{\varepsilon}_t Q_p^{-1}(t)$ *of* $f^Q(x_t) - f^Q(x_{t+1})$ *is evaluated as follows:*

$$\bar{\varepsilon}_t Q_p^{-1} = (\varepsilon_{t+1} - \varepsilon_t) \cdot (x_{t+1} - x_t) \cdot v_t \cdot \tilde{Q}_p^{-1} \tag{39}$$

, where v_t *is a normalized vector defined as* $v_t = \frac{x_{t+1}-x_t}{\|x_{t+1}-x_t\|}$ *and* $\tilde{Q}_p(t)$ *is a scaled quantization parameter to* $Q_p(t)$ *with a constant value* $C \in \mathbf{R}^+$ *such that*

$$\tilde{Q}_p^{-1}(t) = C \cdot b^{\bar{h}(t)}. \tag{40}$$

Proof According to the assumption, we can consider the case represented in the figure 1 as follows:

$$0 = f^Q(x_{t+1}) - f^Q(x_t) = f(x_{t+1}) - f(x_t) + (\varepsilon_{t+1} - \varepsilon_t)Q_p^{-1} \implies f(x_{t+1}) - f(x_t) = -(\varepsilon_{t+1} - \varepsilon_t)Q_p^{-1}. \tag{41}$$

Furthermore, considering the line across the points $(x_t, f(x_t))$ and $(x_{t+1}, f(x_{t+1}))$, we get the following equation for such line:

$$\bar{f}'(x) = \frac{f(x_{t+1}) - f(x_t)}{\|x_{t+1} - x_t\|} v_t \cdot (x - x_t) + f(x_t), \quad \because v_t = \frac{x_{t+1} - x_t}{\|x_{t+1} - x_t\|}. \tag{42}$$

By definition of the quantization parameter, we note that $Q_p(t) = \eta \cdot b^{\bar{h}(t)}$. In addition, Theorem 2 represents $\eta = b^{-\lfloor \log_b(f(x_0)+1) \rfloor}$. Without losing generality, we can set η as follows:

$$\eta = b^{-\lfloor \log_b(f(x_0) - f(x^*))+1 \rfloor}, \quad \forall x \in \mathbf{R}^n, f(x^*) < f(x). \tag{43}$$

Practically, we cannot know the optimal point correctly in most optimization problems, so the above definition for η is an ideal and theoretical case. Letting $Q_p^{-1} = \eta^{-1} \cdot b^{-\bar{h}(t)}$, we obtain

$$Q_p^{-1} = b^{\log_b(f(x_0) - f(x^*))+\epsilon} \cdot b^{\bar{h}(t)} = (f(x_0) - f(x^*)) \cdot b^{\bar{h}(t)+\epsilon} \tag{44}$$

, where ϵ is an error led by the floor operation. Spanning $f(x_0) - f(x^*)$ for a finite value $n > n_0$, we get

$$\begin{aligned} f(x_0) - f(x^*) &= f(x_0) - f(x_1) + f(x_1) - f(x_2) \cdots + f(x_{t+n}) - f(x^*) \\ &\leq \|f(x_0) - f(x_1)\| + \|f(x_1) - f(x_2)\| \cdots + \|f(x_{t+n-1}) - f(x^*)\| \end{aligned} \tag{45}$$

Since Theorem 3 is hold, we can rewrite the final term of the right side such that

$$\begin{aligned} f(x_0) - f(x^*) &\leq \|f(x_0) - f(x_1)\| + \|f(x_1) - f(x_2)\| \cdots + \|f(x_{t+n}) - f(x_{t+n-1})\| \\ &\leq n \cdot \|f(x_{\tau+1}) - f(x_\tau)\| \end{aligned} \tag{46}$$

, where $\tau \in \mathbf{Z}^+$ is defined as

$$\forall t > t_0, \exists \tau \in \mathbf{Z}[t_0, t] \text{ such that } \|f(x_{\tau+1}) - f(x_\tau)\| > \|f(x_{t+1}) - f(x_t)\|. \tag{47}$$

By Lipschitz Continuous, we note $\|f(x_{\tau+1}) - f(x_\tau)\| < L\|x_{\tau+1} - x_\tau\|$, so that we can obtain the following inequality:

$$f(x_0) - f(x^*) < n \cdot L \cdot \|x_{\tau+1} - x_\tau\| \quad \because \tau \in \mathbf{Z}^+. \tag{48}$$

Thus, we can partition the Q_p as follows:

$$Q_p^{-1} = (f(x_0) - f(x^*)) \cdot b^{\bar{h}(t)+\epsilon} < \|x_{\tau+1} - x_\tau\| \cdot n \cdot L \cdot b^{\bar{h}(t)+\epsilon} = \|x_{\tau+1} - x_\tau\| \cdot \bar{Q}_p^{-1} \tag{49}$$

, where $\bar{Q}_p^{-1} \triangleq C_0 \cdot b^{\bar{h}(t)+\epsilon}$ and $C_0 = nL$. Since $\|x_{\tau+1} - x_\tau\| > \|\|x_{t+k+1} - x_{t+k}\|, \forall k \in \mathbf{Z}[1, n]$, we can set a positive value $p > 1$ such that

$$\|x_{\tau+1} - x_\tau\| = p \cdot \|x_{t+1} - x_t\| \tag{50}$$

, for an arbitrary $t > 0$. By (41) and (42), we get

$$f(x_{t+1}) - f(x_t) = -(\varepsilon_{t+1} - \varepsilon_t) Q_p^{-1} \tag{51}$$

and

$$f(x_{t+1}) - f(x_t) = \frac{f(x_{t+1}) - f(x_t)}{\|x_{t+1} - x_t\|} v_t \cdot (x_{t+1} - x_t). \tag{52}$$

Therefore,

$$\begin{aligned} f(x_{t+1}) - f(x_t) &= -(\varepsilon_{t+1} - \varepsilon_t) Q_p^{-1} \\ &= -(\varepsilon_{t+1} - \varepsilon_t) \cdot \|x_{\tau+1} - x_\tau\| \cdot \bar{Q}_p^{-1} \\ &= -(\varepsilon_{t+1} - \varepsilon_t) \cdot \|x_{\tau+1} - x_\tau\| v_t \cdot v_t \cdot \bar{Q}_p^{-1} \\ &= -(\varepsilon_{t+1} - \varepsilon_t) \cdot \|x_{\tau+1} - x_\tau\| \frac{(x_{t+1} - x_t)}{\|x_{t+1} - x_t\|} \cdot v_t \cdot \bar{Q}_p^{-1} \\ &= -(\varepsilon_{t+1} - \varepsilon_t) \cdot \|x_{t+1} - x_t\| \frac{(x_{t+1} - x_t)}{\|x_{t+1} - x_t\|} \cdot v_t \cdot p\bar{Q}_p^{-1} \\ &= -(\varepsilon_{t+1} - \varepsilon_t) \cdot (x_{t+1} - x_t) \cdot v_t \cdot \tilde{Q}_p^{-1} \end{aligned} \tag{53}$$

, where $\tilde{Q}_p^{-1} = p\, C_0\, b^{\bar{h}(t)+\epsilon}$. Consequently, let $\bar{\varepsilon}_t \triangleq \varepsilon_t - \varepsilon_{t-1}$, we can obtain

$$\bar{\varepsilon}_t Q_p^{-1} = (\varepsilon_{t+1} - \varepsilon_t) \cdot (x_{t+1} - x_t) \cdot v_t \cdot \tilde{Q}_p^{-1} \tag{54}$$

■

Theorem 5 *For a given objective function $f(x_t) \in \mathbf{R}$, suppose that there exist the quantized objective functions $f^Q(x_t)$, $f^Q(x_{t+1})$ at a current state x_t and the following state x_{t+1} such that $f^Q(x_t) \geq f^Q(x_{t+1})$, for all $x_{t+1} \neq x_t$; we can obtain the differential equation of the state transition as follows:*

$$dX_t = -\nabla_x f(X_t) dt + \sqrt{C_q} \cdot Q_p^{-1}(t) dW_t \tag{55}$$

where W_t is a standard Wiener process, which has a zero mean and variance with one, X_t is a random variable corresponding to x_t, and $C_q \in \mathbf{R}$ is a constant value.

Proof By definition 1, we can write the quantized objective function $f^Q(x_t)$ as follows:

$$f^Q(x_t) = f(x_t) + \varepsilon_t \cdot Q_p^{-1}(t) \tag{56}$$

According to (56), we can write the difference of the quantized objective function as follows:

$$f^Q(x_{t+1}) - f^Q(x_t) = f(x_{t+1}) - f(x_t) + (\varepsilon_{t+1} - \varepsilon_t) \cdot Q_p^{-1}(t) \tag{57}$$

By the definition of Taylor expansion, we get

$$f(x_{t+1}) - f(x_t) = \nabla_x f(x_t)(x_{t+1} - x_t) + \int_0^1 (1-s) \frac{\partial^2 f}{\partial x^2}(x_t + s(x_{t+1} - x_t))(x_{t+1} - x_t)^2 ds. \tag{58}$$

Accordingly, calculating (57), we can obtain

$$0 = f^Q(x_{t+1}) - f^Q(x_t) = f(x_{t+1}) - f(x_t) + \bar{\varepsilon}_t \cdot Q_p^{-1}(t) \Rightarrow f(x_{t+1}) - f(x_t) = -\bar{\varepsilon}_t \cdot Q_p^{-1}(t) \tag{59}$$

, where $\bar{\varepsilon}_t$ is a difference of ε_t such that $\bar{\varepsilon}_t \triangleq \varepsilon_{t+1} - \varepsilon_t$ and $\bar{\varepsilon}_t \in \mathbf{R}[-1, 1]$. Since Lipschitz continuous condition (Assumption 2) and $f^Q(x_{t+1}) - f^Q(x_t) \leq 0$ is hold, there exists a positive value $m \in \mathbf{R}^+$ such that

$$m \triangleq \inf_x \left| \frac{\partial^2 f}{\partial x^2}(x) \right|, \forall x = x_t + s(x_{t+1} - x_t),\ s \in \mathbf{R}[0, 1]. \tag{60}$$

Using (54) and (60), we rewrite (58) as follows:

$$\begin{aligned} f(x_{t+1}) - f(x_t) &> \nabla_x f(x_t)(x_{t+1} - x_t) + m(x_{t+1} - x_t)^2 \int_0^1 (1-s)ds + \bar{\varepsilon}_t Q_p^{-1}(t) \\ &= (x_{t+1} - x_t) \cdot \nabla_x f(x_t) + \frac{m}{2}(x_{t+1} - x_t)^2 + \bar{\varepsilon}_t Q_p^{-1}(t) \\ &= -(x_{t+1} - x_t) \cdot \left(-\nabla_x f(x_t) + v_t \cdot \bar{\varepsilon}_t \tilde{Q}_p^{-1}(t) \right) + \frac{m}{2}(x_{t+1} - x_t)^2 \end{aligned} \tag{61}$$

, where v_t is a normalized vector such that $v_t = \frac{(x_{t+1} - x_t)}{\|x_{t+1} - x_t\|}$. In (61), if we choose $(x_{t+1} - x_t)$ appropriately, we note that there exist a positive m satisfying the inequality condition $f(x_{t+1}) \leq f(x_t)$. Thereby, when we set $x_{t+1} - x_t$ as follows

$$x_{t+1} - x_t = -\nabla_x f(x_t) + v_t \cdot \bar{\varepsilon}_t \tilde{Q}_p^{-1}(t) \tag{62}$$

, we can obtain the following inequality:

$$0 \geq f(x_{t+1}) - f(x_t) > (x_{t+1} - x_t)^2 \left(\frac{m}{2} - 1 \right). \tag{63}$$

Consequently, when the infimum to the second derivation of the objective function $f(x)$ fulfills $0 \leq m < 2$, we can find the state x_{t+1} satisfying the inequality $f(x_{t+1}) - f(x_t)$. Conversely, if $m > 2$, it contradicts $f(x_{t+1}) \leq f(x_t)$. In other words, (63) turns to the follwoing inequality:

$$f(x_{t+1}) - f(x_t) > (x_{t+1} - x_t)^2 \left(\frac{m}{2} - 1 \right) > 0,\ \forall m > 2. \tag{64}$$

(64) implies $f(x_{t+1}) > f(x_t)$, and it means that the proposed algorithm brings a hill-climbing effect within the domain fillfills the quantized range such as $x \in \{x | f^Q(x) = f^Q(x_{t+1}) = f^Q(x_t)\}$. Since the proposed algorithm serves the bounded range provided by the quantization for each iteration, the hill-climbing effect cannot lead to divergence of the algorithm.

To obtain a differential form of the difference to x_t, we let $Z(s) = x_t + s(x_{t+1} - x_t)$ and rewrite (62) as following integral equation:

$$x_{t+1} - x_t = (x_{t+1} - x_t) \int_0^1 ds = \int_0^1 (x_{t+1} - x_t) ds = \int_0^1 \frac{\partial Z(s)}{\partial s} ds = \int_0^1 dZ(s). \tag{65}$$

From (62), we can get

$$x_{t+1} - x_t = \int_0^1 dZ(s) = \int_0^1 (x_{t+1} - x_t) ds = \int_0^1 (-\nabla_x f(x_t) + v_t \cdot \bar{\varepsilon}_t \tilde{Q}_p^{-1}(t)) ds = \int_t^{t+1} dx_s. \tag{66}$$

Herein, since v_t is a normalized vector, we can get the variance of $v_t \cdot \bar{\varepsilon}_t \tilde{Q}_p^{-1}(t)$ such that

$$\mathbb{E}_{\mathcal{F}_t} \langle v_t, v_t \rangle \cdot \bar{\varepsilon}_t^2 \tilde{Q}_p^{-2}(t) = \tilde{Q}_p^{-2}(t) \mathbb{E}_{\mathcal{F}_t} \bar{\varepsilon}_t^2 = \frac{4}{12 \cdot \tilde{Q}_p^2(t)} = C_q \tilde{Q}_p^{-2}(t) \quad \because \|v_t\| = 1,\ C_q = 1/3 \tag{67}$$

Differentiating the two right-most terms in (66), we obtain

$$\begin{aligned} \frac{\partial}{\partial s} \int dx_s \Big|_{s=t} &= \frac{\partial}{\partial s} \int (-\nabla_x f(x_t) + v_t \cdot \bar{\varepsilon}_t \tilde{Q}_p^{-1}(t)) ds \Big|_{s=t} \\ \implies dX_t &= -\nabla_x f(X_t) dt + v_t \cdot \bar{\varepsilon}_t \tilde{Q}_p^{-1}(t) dt \\ \implies dX_t &= -\nabla_x f(X_t) dt + \sqrt{C_q} \cdot \tilde{Q}_p^{-1} dW_t \end{aligned} \tag{68}$$

■

Theorem 5 gives the fundamental stochastic differential form to evaluate the optimal quantization schedule for global optimization.

In (63), you can argue that, if m is larger than two, then the inequality is broken. However, since m is just an infimum of the second derivation of the objective function, not the correct value, we can regard it as a quadratic approximated function to the objective function. Thus, the proposition holds sufficiently when the objective function is locally convex on some domain around x_t. The more important point is that the proposition also holds when m is negative or zero. Negative m is that the objective function is a concave function on a local domain of x_t. In a conventional convex optimization theory, we cannot obtain a less value of an objective function at a next state x_{t+1} based on a negative gradient than a current value. However, since the proposed algorithm can get a lower value of the objective function at the next state despite a concave function, m can be equal to or less than zero. Additionally, when the value of the objective function on the next state is larger than the current state, the quantization makes the next and the current value of the objective function equal so that the proposition still holds.

Even though the proposed algorithm does not have any scheduler like the temperature scheduler in simulated annealing, if the quantization parameter decreases to the schedule provided by the following proposition, the proposed algorithm can find the global optimum in the minima possibly found.

Theorem 6 *If the dynamics of the state transition by the proposed algorithm follow (14), the state x_t weakly converges to the global minimum when the quantization parameter decreases to the following schedule:*

$$\inf_{t \geq 0} Q_p^{-1}(t) = \frac{\sqrt{12} \cdot C}{\log(t+2)}, \quad C \in \mathbf{R}^+,\ C \gg 0 \tag{69}$$

Proof For the proof of the theorem, we depend on the lemmas in works of Geman and Hwang [13]. First, we prove the following convergence of the transition probability:

$$\lim_{\tau \to \infty} \sup_{x_t, x_{t+\tau} \in \mathbf{R}^n} \| p(t, \bar{x}_t, t+\tau, x^*) - p(t, x_t, t+\tau, x^*) \| = 0 \tag{70}$$

, where t and τ is the current time index and the process time index, respectively. x^* represents an global optimum for the objective function $f(x_t)$.

Let the infimum of the transition probability from t to $t+1$ such that

$$\delta_t = \inf_{x,y \in \mathbf{R}^n} p(t, x, t+1, y) \tag{71}$$

According to the lemma in Geman and Hwang [13], the upper bound of (70) is

$$\overline{\lim_{\tau \to \infty}} \sup_{x_t, x_{t+\tau} \in \mathbf{R}^n} \| p(t, \bar{x}_t, t+\tau, x^*) - p(t, x_t, t+\tau, x^*) \| \leq 2\|x^*\|_\infty \prod_{k=0}^{\infty} (1 - \delta_{t+k}). \tag{72}$$

From the exponential approximation (1) in Lemma:Auxiliary, we rewrite (72) as follows:

$$\overline{\lim_{\tau \to \infty}} \sup_{x_t, x_{t+\tau} \in \mathbf{R}^n} \| p(t, \bar{x}_t, t+\tau, x^*) - p(t, x_t, t+\tau, x^*) \| \leq 2\|x^*\|_\infty \exp(-\sum_{k=0}^{\infty} \delta_{t+k}). \tag{73}$$

Herein, to obtain the bound of δ_{t+k}, we rewrite the SDE for the dynamics of the proposed algorithm from Theorem 6:

$$dX_s = -\nabla f(X_s) ds + \sigma(s) \sqrt{C_q} dW_s, \quad s \in \mathbf{R}(t, t+1). \tag{74}$$

, where $\sigma(s) \triangleq Q_p^{-1}(s)$.

Define a domain $\mathcal{F}\{f : [t, t+1] \to \mathbf{R}^n, f \text{ continuous }\}$, Let P_x be the probability measures on $\mathcal{F}$ induced by (74) and the probability distribution Q_x given by the following equation:

$$d\bar{X}_s = \sigma(s) \sqrt{C_q} dW_s, \quad s \in \mathbf{R}(t, t+1). \tag{75}$$

According to the Girsanov theorem (Klebaner [30], Øksendal [44]), we obtain

$$\frac{dP_x}{dQ_x} = \exp\left\{ -\int_t^{t+1} \frac{C_q^{-1}}{\sigma^2(s)} \nabla_x f(X_s) d\bar{X}_s - \frac{1}{2} \int_t^{t+1} \frac{C_q^{-1}}{\sigma^2(s)} \|\nabla_x f(X_s)\|^2 ds \right\}. \tag{76}$$

To compute the upper bound of (76), we will check the upper bound of $\|\nabla_x f\|$. Considering Assumption 2, the gradient of $f(x_t) \in C^2$ fulfills the Lipschitz continuous condition as well. Thereby, there exist a positive value L' such that

$$\|\nabla f(w_s) - \nabla f(x^*)\| \leq L' \|w_s - x^*\|, \quad \forall s > 0. \tag{77}$$

Successively, since $\nabla_x f(x^*) = 0$, the Lipschitz condition forms simply as follows :

$$\|\nabla_x f(x_t)\| \leq L' \rho = C_0 \tag{78}$$

, where $\rho = \|x_t - x^*\|$.

Consequently, for all $s \in \mathbf{R}[t, t+1)$, we compute the upper bound of the first term in exponential function in (76) as follows:

$$\begin{aligned} \left\| \int_t^{t+1} \frac{C_q^{-1}}{\sigma^2(s)} \nabla_x f(X_s) d\bar{X}_s \right\| &\leq \int_t^{t+1} \left\| \frac{C_q^{-1}}{\sigma^2(s)} \nabla_x f(X_s) d\bar{X}_s \right\| \\ &\leq \int_t^{t+1} \frac{C_q^{-1}}{\sigma^2(s)} \|\nabla_x f(X_s)\| \sigma(s) \sqrt{C_q} dW_s \\ &\leq \frac{\sqrt{C_q^{-1}}}{\sigma(s)} \sup \|\nabla_x f(X_s)\| \int_t^{t+1} dW_s \\ &\leq \frac{\sqrt{C_q^{-1}}}{\sigma(s)} C_0 \|W_t - \frac{1}{2}\| \leq \frac{1}{\sigma(s)} C_0 \sqrt{C_q^{-1}} (\rho + \frac{1}{2}). \end{aligned} \tag{79}$$

(79) implies that

$$\left\| -\int_t^{t+1} \frac{C_q^{-1}}{\sigma(s)} \nabla_x f(X_s) d\bar{X}_s \right\| \leq \frac{C_1}{\sigma(s)} \tag{80}$$

, where C_1 is positive value such that $C_1 > C_0\sqrt{C_q^{-1}}(\rho + \frac{1}{2})$.

In addition, the upper bound of the second term is

$$\begin{aligned}\frac{1}{2}\left\| \int_t^{t+1} \frac{C_q^{-1}}{\sigma^2(s)} \|\nabla_x f(X_s)\|^2 ds \right\| &\leq \frac{1}{2}\int_t^{t+1} \frac{C_q^{-1}}{\sigma^2(s)} \|\nabla_x f(X_s)\|^2 ds \\ &\leq \frac{1}{2}\frac{C_q^{-1}}{\sigma^2(s)} \sup \|\nabla_x f(X_s)\|^2 \int_t^{t+1} ds \\ &\leq \frac{1}{2\sigma^2(s)} C_q^{-1} \cdot C_0^2 \leq \frac{C_2}{2\sigma^2(s)}, \quad \because C_2 > C_q^{-1} \cdot C_0^2.\end{aligned} \tag{81}$$

Since $\sigma(s) \triangleq Q_p^{-1}(t)$ is monotone decreasing function, the supremum of $\sigma(s)$ is $\sigma(0)$ for all $s \in \mathbf{R}[0, \infty)$, i.e. $\sup_{s \in \mathbf{R}[0,\infty]} \sigma(s) = \sigma(0) \triangleq \sigma$. With the supremum of each term in (76), we can obtain the lower bound of the Radon-Nykodym derivative (76) such that

$$\frac{dP_w}{dQ_w} \geq \exp\left(-\frac{1}{\sigma(s)}\left(C_1 + \frac{C_2}{2\sigma(s)}\right)\right) \geq \exp\left(-\frac{C_3}{\sigma(s)}\right), \quad \because C_3 > 2\sigma(0)C_2 + C_1. \tag{82}$$

Accordingly, for any $\varepsilon > 0$ and $x_t,\ x^* \in \mathbf{R}^n$, the infimum of $P_x(|X_{t+1} - x^*| < \varepsilon)$ is

$$P_x(|X_{t+1} - x^*| < \varepsilon) \geq \exp\left(-\frac{C_3}{\sigma(s)}\right) Q_x(|X_{t+1} - x^*| < \varepsilon). \tag{83}$$

Since Q_w is a normal distribution based on (75), we have

$$\begin{aligned}P_x(|X_{t+1} - x^*| < \varepsilon) &\geq \exp\left(-\frac{C_3}{\sigma(s)}\right) \int_{\|x-x^*\|<\varepsilon} \frac{1}{\sigma\sqrt{2\pi \int_t^{t+1} C_q d\tau}} \exp\left(-\frac{(x-x^*)^2}{2\int_t^{t+1} C_q d\tau}\right) dx \\ &\geq \exp\left(-\frac{C_3}{\sigma(s)}\right) \int_{\|x-x^*\|<\varepsilon} \frac{1}{\sigma\sqrt{2\pi C_q \int_t^{t+1} d\tau}} \exp\left(-\frac{(\sqrt{\rho}+\varepsilon)^2}{2C_q \int_t^{t+1} d\tau}\right) dx \\ &\geq \exp\left(-\frac{C_3}{\sigma(s)}\right) \frac{1}{\sigma(0)\sqrt{2\pi C_q}} \exp\left(-\frac{(\sqrt{\rho}+\varepsilon)^2}{2C_q}\right) \int_{\|x-x^*\|<\varepsilon} dx \\ &= \exp\left(-\frac{C_3}{\sigma(s)}\right) \frac{1}{\sigma(0)\sqrt{2\pi C_q}} \exp\left(-\frac{(\sqrt{\rho}+\varepsilon)^2}{2C_q}\right) 2\varepsilon \\ &\geq \exp\left(-\frac{C_3}{\sigma(s)}\right) \frac{1}{\sigma(0)\sqrt{2\pi C_q}} \left(1 + \frac{(\sqrt{\rho}+\varepsilon)^2}{2C_q}\right) 2\varepsilon \\ &\geq \exp\left(-\frac{C_3}{\sigma(s)}\right) \frac{1}{\sigma(0)\sqrt{2\pi C_q}} \left(\frac{2C_q + (\sqrt{\rho}+\varepsilon)^2}{C_q}\right)\Bigg|_{\rho=0,\varepsilon=0} \cdot \varepsilon \\ &\geq \exp\left(-\frac{C_3}{\sigma(s)}\right) \cdot C_4 \cdot \varepsilon, \quad \because C_4 = \frac{\sqrt{2}}{\sigma(0)\sqrt{\pi C_q}}.\end{aligned} \tag{84}$$

Finally, we obtain the lower bound of the transition probability such that

$$
\begin{aligned}
\delta_t &= \inf_{x,y \in \mathbf{R}^n} p(t, x, t+1, y) \Big|_{x=x_t,\ y=x^*} \\
&= \inf_{x,y \in \mathbf{R}^n} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} P_x(|X_{t+1} - x^*| < \varepsilon) \\
&\geq \inf_{x,y \in \mathbf{R}^n} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \cdot C_4 \cdot \exp\left(-\frac{C_3}{\sigma(t)}\right) \cdot \cdot \varepsilon \\
&\geq \exp\left(-\frac{C_5}{\sigma(t)}\right), \quad \because C_5 > C_3 + \sigma(0) \cdot |\ln C_4|
\end{aligned}
$$

The above inequality implies that, if there exists a monotone decreasing function such that $\sigma(s) \geq \frac{C_5}{\log(t+2)}$, it satisfies that the convergence condition given by (73) such that

$$
\sum_{k=0}^{\infty} \delta_{t+k} \geq \sum_{k=0}^{\infty} \exp\left(-\frac{C_5}{C_5} \log(t+2+k)\right) = \sum_{k=0}^{\infty} \frac{1}{t+2+k} = \infty, \quad \forall k \geq 0. \tag{85}
$$

Substitute (85) into (73), we obtain

$$
\overline{\lim_{\tau \to \infty}} \sup_{x_t, x_{t+\tau} \in \mathbf{R}^n} \|p(t, \bar{x}_t, t+\tau, x^*) - p(t, x_t, t+\tau, x^*)\| \leq 2\|x^*\|_\infty \exp(-\sum_{k=0}^{\infty} \delta_{t+k})) = 0. \tag{86}
$$

■

E.3. Proof of theorem 4.3

Although Theorem 6 provides the scheduler of the quantization parameter obtaining the global minimum, the scheduler is not practical. Whereas the quantization parameter is a rational number, the value of the scheduler is a real number. Therefore, we have to set an appropriate bound of the scheduler for the quantization parameter. The following theorem gives one instance.

Theorem 7 *Suppose that there exists an integer valued annealing schedule $\sigma(t) \in \mathbf{Z}^+$ such that $\sigma(t) \geq \inf \sigma(t) \triangleq c/\log(t+2)$. If the power function $\bar{h}(t)$ of the quantization parameter $Q_p^{-1}(t)$ fulfills the following condition, the proposed algorithm weakly converges to the global optimum.*

$$
\log_b \left(C_0 \cdot b^{-\frac{2\beta}{t+2}} \cdot \inf \sigma(t)\right) \leq \bar{h}(t) \leq \log_b \left(C_1 \log(t+2)\right) \tag{87}
$$

, where $C_0 \equiv \eta\sqrt{C_q}$ and $C_1 \equiv \sqrt{C_q}\eta/C$.

Proof From the Theorem 5, 6, we obtain the infimum of $\sigma(t) \triangleq \sqrt{C_q} Q_p^{-1}(t)$. To evaluate the integer value of the quantization resolution $Q_p(t)$, we set $T(t)$ to be a supremum of $\sigma(t)$ such that

$$
\frac{C}{\log(t+2)} \leq \sigma(t) \leq T(t). \tag{88}
$$

In (88), $T(t)$ is a monotone decreasing function, such as $T(t) \downarrow 0$ with respect to $t \uparrow 0$. Moreover, when Δ is given as $\Delta \equiv \sup_{x,y \in \mathbf{R}}(f(x) - f(y)$, $T(t)$ includes the following properties:

$$\frac{d}{dt} \exp\left(-\frac{2\Delta}{T(t)}\right) = \frac{dT(t)}{dt} \cdot \frac{1}{T^2(t)} \exp\left(-\frac{2\Delta}{T(t)}\right) \to 0, \quad \text{as } t \uparrow \infty \tag{89}$$

From Definition 2, we note $Q_p = \eta \cdot b^{-\bar{h}(t)}$, so that we substitute $\sigma(t)$ with $Q_p(t)$ in (88), as follows:

$$\frac{C}{\log(t+2)} \leq \sqrt{C_q} \cdot \eta \cdot b^{-\bar{h}(t)} \leq T(t). \tag{90}$$

Applying the log function to each term and rearranging, we obtain

$$\log_b\left(\frac{\sqrt{C_q}\eta}{T(t)}\right) \leq \bar{h}(t) \leq \log_b\left(\frac{\sqrt{C_q}\eta \cdot \log(t+2)}{C}\right). \tag{91}$$

Let $T(t) \triangleq b^{\frac{2\beta}{t+2}} \cdot (\inf_{t \geq 0} \sigma(t))^{-1}$, then we get

$$\log_b\left(\eta\sqrt{C_q} \cdot b^{-\frac{2\beta}{t+2}} \inf_{t \geq 0} \sigma(t)\right) \leq \bar{h}(t) \leq \log_b\left(\frac{\sqrt{C_q}\eta \cdot \log(t+2)}{C}\right). \tag{92}$$

Let $C_0 \equiv \eta\sqrt{C_q}$ and $C_1 \equiv \sqrt{C_q}\eta/C$, then theorem holds. ■

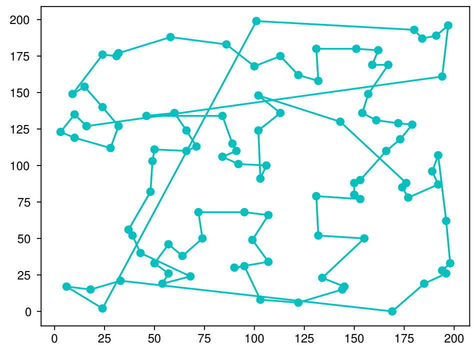
E.4. Travelling Salesman Problem (TSP)

As mentioned in the manuscript, we accomplish the optimization test with equal and fixed locations of cities for all attempts. Figure 2 shows the initial path given by the nearest neighborhood algorithm, the final route given by simulated annealing, quantum annealing, and the proposed optimization algorithm.

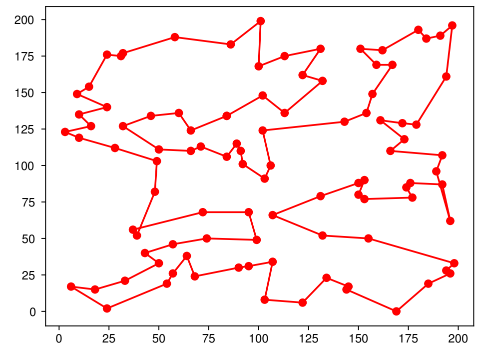
Figure 3 shows the trends of the minimum cost produced by each tested algorithm. Since simulated annealing and quantum annealing employ an acceptance probability, the trends of the two algorithms represent fluctuation in the early stage of optimization. However, the proposed algorithm does not include acceptance probability, so that the minimum cost decreases with relatively small fluctuation seen in simulated annealing and quantum annealing.

In addition, such slight fluctuation brings a fast convergence to a feasible (or global) solution compared to other algorithms. The quantization employed in the proposed algorithm provides a hill-climbing effect as other algorithms do. However, the proposed algorithm suppresses the hill climbing effect reasonably so that the candidates provided by the optimization algorithm cannot diverge to an unfeasible solution so far. On the other hand, the other algorithms permit the candidate to diverge under the acceptable probability, requiring more time to converge, even if the algorithm can find the global minima.

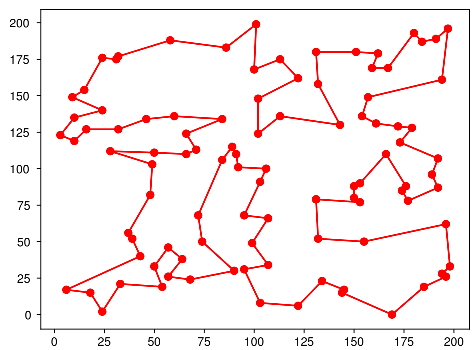
The reasonable hill-climbing provided by the proposed algorithm represents robust optimization performance compared to other algorithms using an acceptance probability.



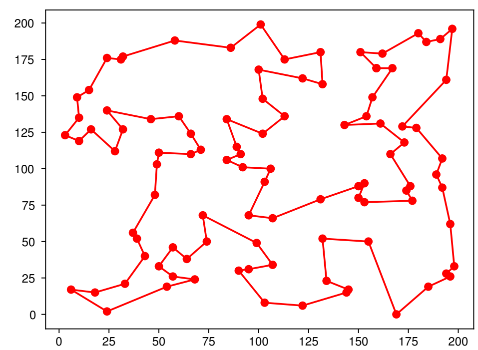
(*a*) Initial path given by the nearest neighborhood algorithm (cost is 2159)



(*b*) Final path given by the simulated annealing algorithm (the minimum cost is 1731)



(*c*) Final path given by the quantum annealing algorithm (the minimum cost is 1706)



(*d*) Final path given by the proposed algorithm (the minimum cost is 1636)

Figure 2: Comparison of TSP routes provided by each optimization algorithm

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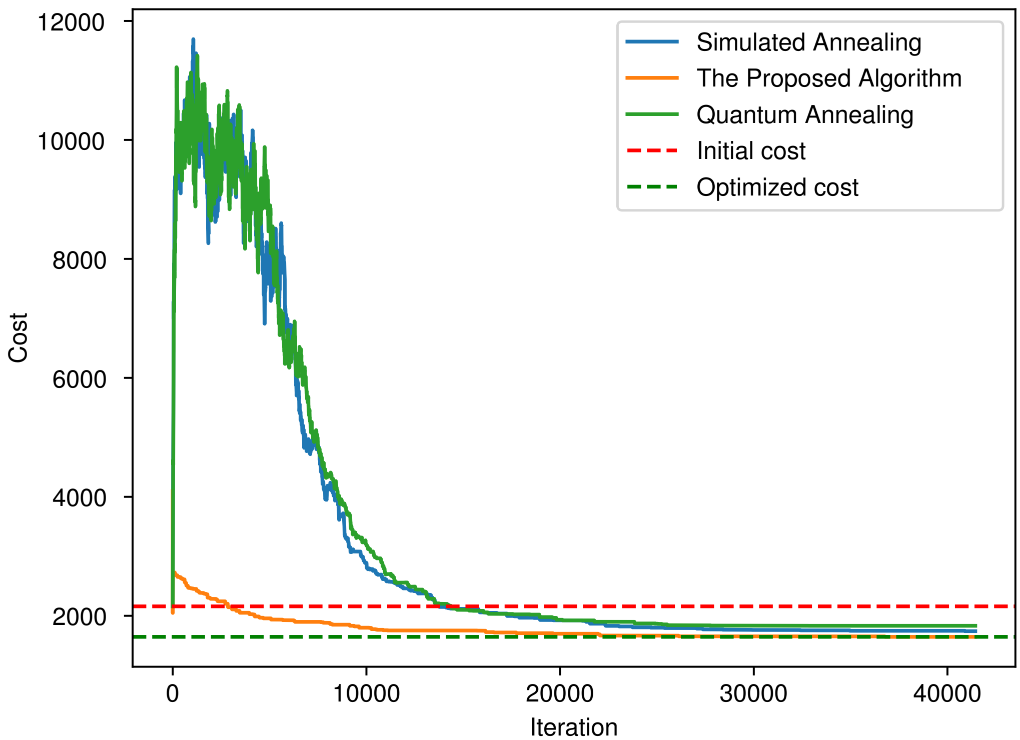

Figure 3: The minimum cost trends corresponding to each algorithm to iterations in the TSP optimization test

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